

On quark masses in holographic QCD

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ABSTRACT: Recently certain nonlocal operators were proposed to provide quark masses for the holographic model of QCD developed by Sakai and Sugimoto. The properties of these operators at strong coupling are examined in detail using holographic techniques. We find the renormalization procedure for these operators is modified by the running of the five-dimensional gauge coupling. We explicitly evaluate the chiral condensate characterized by these operators.

KEYWORDS: Gauge-gravity correspondence, Spontaneous Symmetry Breaking, Intersecting branes models.

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1. Introduction

Gauge/gravity dualities have proven to be a remarkable new framework to study a large class of strongly coupled gauge theories [1, 2]. However, the gauge theories that are currently amenable to such holographic analysis are typically very different from real world QCD. Hence constructing a holographic model of QCD remains one of the most important challenges for this approach. Currently, the most successful proposal is a construction by Sakai and Sugimoto [3, 4] based on a configuration of D8- and $\overline{\text{D8}}$ -branes in a D4-brane background. While reliable calculations are limited to large N_c and small N_f/N_c , many observables seem to show a good approximation to real QCD at low energies.

A key feature of the Sakai-Sugimoto model is that it exhibits the desired non-Abelian chiral symmetry $U(N_f)_L \times U(N_f)_R$, as well as its spontaneous breaking [3, 4]. Of course, in real world QCD, the analogous symmetry is only approximate as it is explicitly broken by the quark masses. A shortcoming of the D8/ $\overline{\text{D8}}$ /D4 model then is that the quarks are precisely massless. While various suggestions have been made to introduce quark masses [5–9], there remain technical difficulties in pursuing these proposals in detail. A recent proposal which seems easier to study is based on deforming the model with certain nonlocal operators [10–12]. The underlying microscopic field theory is a five-dimensional gauge theory where the chiral quarks are localized on separate four-dimensional defects. Since the fermions of different chiralities are separated in the five-dimensional spacetime,

no simple local mass term can be introduced in the UV field theory. However, this spatial separation can be overcome by connecting two quark fields with a Wilson line. Hence a natural suggestion is to introduce a nonlocal operator to provide a quark mass deformation [10–12]:

$$[\psi_L(x^\mu, x^4 = 0)]^a P \exp \left[i \int_{-L/2}^{L/2} A_4 dx^4 \right]_a^b [\psi_R(x^\mu, x^4 = L)]_b. \quad (1.1)$$

As has been extensively studied for closed Wilson lines [13, 14], such a nonlocal operator would be dual to an instantonic string worldsheet which extends between the D8- $\overline{\text{D8}}$ pair. In the following, we examine the properties of these operators in some detail.

An overview of the paper is as follows: in section 2, we review the construction of the Sakai-Sugimoto background. In section 3, we consider the nonlocal mass terms introduced in [10, 11]. In particular, we examine the affect of the dilaton coupling to the string worldsheet. Even though this coupling only appears at higher order in the α' expansion, we find that in the D4-brane background it introduces a interesting modification in the renormalization of the Wilson line. In section 4, we explicitly calculate the expectation value of these nonlocal operators. In the absence of any local fermion bilinears, this expectation value or condensate is an order parameter characterizing the chiral symmetry breaking in this holographic model. We close in section 5 with a discussion of our results and by making a few observations about future directions. Appendix A provides the details of a calculation of the fluctuation determinant of the worldsheet fields. The latter contributes at the same order as the dilaton coupling but does not make any further modifications of the renormalization of the nonlocal operators.

2. Review of Sakai-Sugimoto background

The Sakai-Sugimoto model [3, 4] is based on the throat limit of intersecting D4- and D8-branes, summarized by the array

$$\begin{array}{cccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 \text{D4} : & \times & \times & \times & \times & \times & & & & & \\
 \text{D8} : & \times & \times & \times & \times & & \times & \times & \times & \times & \times \\
 \overline{\text{D8}} : & \times & \times & \times & \times & & \times & \times & \times & \times & \times
 \end{array} . \quad (2.1)$$

The world-volume theory of the N_c D4-branes naturally gives rise to a maximally supersymmetric $U(N_c)$ gauge theory in five dimensions. Following [15], the x^4 direction is compactified and antiperiodic boundary conditions are imposed on the fermionic fields around this circle. In the far infrared, one might expect that the only relevant degrees of freedom arising from this D4 world-volume theory correspond to four-dimensional Yang-Mills with gauge group $SU(N_c)$. Further the intersection of the D4-branes with N_f D8-branes supports chiral fermions in the fundamental representations of the gauge group $U(N_c)$ and of the $U(N_f)$ flavour symmetry. These fermions propagate in the $3 + 1$ dimensions common to both sets of branes. Similarly, the intersection with N_f $\overline{\text{D8}}$ -branes produces an analogous

set of four-dimensional anti-chiral fermions. Hence, the Sakai-Sugimoto model produces a holographic description of QCD in the throat limit of the D4-branes. The dual gravity theory in this framework yields reliable results for large N_c and strong 't Hooft coupling. Our current understanding of this holographic model is limited to the quenched approximation, i.e., $N_f/N_c \rightarrow 0$, in which the D8-branes are probes in the supergravity background.

2.1 D4-brane background

Here we review the supergravity background, which we refer to as the the D4 soliton (following the nomenclature of [16]). This throat geometry for a stack of N_c D4-branes with antiperiodic fermions on the x^4 circle is the gravitational dual of a confined phase of the $U(N_c)$ gauge theory [15], as described above. For comparison purposes, we also consider the supersymmetric D4-brane throat with fermions that are periodic on the x^4 circle. Both solutions can be expressed in the form¹

$$ds^2 = \left(\frac{u}{R}\right)^{\frac{3}{2}} \left(-dt^2 + \delta_{ij} dx^i dx^j + f(u)(dx^4)^2\right) + \left(\frac{R}{u}\right)^{\frac{3}{2}} \left(\frac{du^2}{f(u)} + u^2 d\Omega_4^2\right) \quad (2.2)$$

$$e^\phi = g_s \left(\frac{u}{R}\right)^{\frac{3}{4}} \quad F_4 = 3\pi N_c \ell_s^3 \epsilon_4. \quad (2.3)$$

The four noncompact directions of the gauge theory correspond to $t = x^0$ and x^i with $i = 1, 2, 3$, while the coordinate x^4 labels the compact direction. The 56789-directions transverse to the D4-branes are described by a radial coordinate u and four angles that parameterize a unit four-sphere. The $SO(5)$ -invariant line element on this sphere is $d\Omega_4^2$, and the volume form is ϵ_4 . The function $f(u)$ is given by

$$f(u) = 1 - \frac{u_{\text{KK}}^3}{u^3}, \quad (2.4)$$

but the constant $u_{\text{KK}} = 0$ for the supersymmetric background.

The D4 soliton appears to have a conical singularity at $u = u_{\text{KK}}$. Regularity requires that the period of the compact direction, $x^4 \sim x^4 + 2\pi r_4$, is given by

$$2\pi r_4 = \frac{4\pi}{3} \frac{R^{3/2}}{u_{\text{KK}}^{1/2}}. \quad (2.5)$$

With this choice the x^4 circle smoothly shrinks to zero size at $u = u_{\text{KK}}$. Fermionic fields in the bulk must be antiperiodic on this circle, reflecting the antiperiodic boundary condition on fermions in the dual gauge theory. Unlike the soliton background, the supersymmetric D4-brane geometry with $u_{\text{KK}} = 0$ exhibits a naked curvature singularity at $u = 0$. In that case there is no restriction on the periodicity of the x^4 direction. Further, the dual gauge theory is not confining.

The supergravity solution described above is completely specified by the string coupling constant, g_s , the RR flux quantum (i.e., the number of D4-branes), N_c , and the non-extremality constant, u_{KK} . The remaining parameter is a length scale, R , which is given

¹The normalization for F_4 is different from what is prevalent in the literature and has been chosen to be consistent with the usual IIA action, $I = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{g} (e^{-2\phi}(R + 4(\nabla\phi)^2) - \frac{1}{48}F_4^2)$.

in terms of these quantities and the string length, ℓ_s , by

$$R^3 = \pi g_s N_c \ell_s^3 . \tag{2.6}$$

Various combinations of these parameters have direct interpretations in the dual gauge theory. The holographic dictionary gives the five-dimensional gauge coupling as $g_5^2 = (2\pi)^2 g_s \ell_s$, so that the five-dimensional 't Hooft coupling is

$$\lambda_5 = g_5^2 N_c . \tag{2.7}$$

Since these couplings have dimensions of length there is a power-law running of the dimensionless effective coupling [17]

$$g_{\text{eff}}^2 = g_5^2 N_c U , \tag{2.8}$$

where the energy scale U is related to the radial coordinate u in the D4-brane throat by $U = u/2\pi\ell_s^2$ [18]. One finds that the scale of Kaluza-Klein excitations of the compactified coordinate x^4 gives the characteristic mass for glueballs [19]

$$M_{\text{KK}} = \frac{1}{r_4} = \frac{3 u_{\text{KK}}^{1/2}}{2 R^{3/2}} . \tag{2.9}$$

Below this scale, the low-energy gauge coupling in four dimensions is $g_{\text{YM}}^2 = g_5^2/2\pi r_4$.

Supergravity provides a good description of physics in the D4-brane background if two conditions are met. First, gravity calculations are reliable if the length scale associated with spacetime curvatures is small compared to the fundamental string tension. In the D4 soliton solution (2.4) the Ricci scalar has a maximum at $u \sim u_{\text{KK}}$, where curvatures are of order $(u_{\text{KK}} R^3)^{-1/2}$. Hence we require

$$\frac{u_{\text{KK}}^{1/2} R^{3/2}}{\ell_s^2} \gg 1 . \tag{2.10}$$

In terms of gauge theory quantities this can be expressed as the condition

$$g_{\text{YM}}^2 N_c \gg 1 \tag{2.11}$$

so that the restriction to small curvatures corresponds to a large 't Hooft coupling in the effective four-dimensional gauge theory. Second, string loop effects are suppressed as long as the local string coupling is small: $e^\phi \ll 1$. The form of the dilaton (2.3) implies that, for finite values of the gauge theory parameters, the inequality $e^\phi \ll 1$ can only be satisfied over some finite range of the coordinate u . The string coupling eventually becomes $\mathcal{O}(1)$ at a value of u given by

$$u_{\text{crit}} \simeq \frac{N_c^{1/3} \ell_s^2}{g_{\text{YM}}^2 r_4} . \tag{2.12}$$

This critical radius naturally becomes large in the limit $g_{\text{YM}} \rightarrow 0$. Taken together, equations (2.10) and (2.12) indicate that the supergravity analysis in the D4-soliton background is reliable in precisely the strong-coupling regime of the 't Hooft limit of the four-dimensional gauge theory.

In the strong coupling regime the QCD scale cannot be decoupled from the compactification scale, e.g., in the confining phase described by the D4 soliton, the QCD string tension is $T \sim g_{\text{YM}}^2 N_c / r_4^2$ [15, 20]. This means that, for most practical purposes, calculations in the holographic framework are reliable in a regime corresponding to a five-dimensional gauge theory. Since this theory is nonrenormalizable it should be thought of as being defined with a cut-off scale, $U_\infty = u_\infty / 2\pi\alpha'$. Above this scale a UV completion with new degrees of freedom is required. This completion may be a lift to M-theory, with the x^{11} circle opening up to reveal an asymptotically $\text{AdS}_7 \times S^4$ background (with identifications). On the field theory side of the duality, the UV completion of the five-dimensional Yang-Mills theory is given by the six-dimensional (2, 0) theory compactified on a circle. An alternative UV completion would simply be type IIA superstring theory in the asymptotically flat D4-brane background.

2.2 D8-brane embeddings

Our current understanding of the holographic model described in the previous section is largely limited to the quenched approximation: $N_f/N_c \rightarrow 0$. In this limit the D8-branes can be treated as probes embedded in the supergravity background generated by the D4-branes.² The D8-brane fills the noncompact 0123 directions as well as the angles on the S^4 transverse to the D4-branes. The nontrivial aspect of the embedding is given by a function $x^4(u)$ that characterizes the D8-brane's profile in the u - x^4 plane. With this choice of embedding, the action for the D8-branes is

$$I_{\text{D8}} \sim -N_f T_{\text{D8}} \int du u^4 \sqrt{f(u) (\partial_u x^4)^2 + \left(\frac{R}{u}\right)^3 \frac{1}{f(u)}}. \quad (2.13)$$

The resulting equation of motion for $x^4(u)$ is

$$\frac{\partial}{\partial u} \left(\frac{u^4 f(u) \partial_u x^4}{\sqrt{f(u) (\partial_u x^4)^2 + (R/u)^3 / f(u)}} \right) = 0. \quad (2.14)$$

The expression within the parentheses is constant. If we assume that the profile is symmetric across the u -axis, crosses this axis at some value u_0 , and is smooth in the vicinity of this point, then this constant is given by $u_0^4 \sqrt{f(u_0)}$. The embedding equation can then be expressed as

$$\partial_u x^4 = \frac{R^{3/2} u_0^4 \sqrt{f(u_0)}}{u^{3/2} f(u) \sqrt{u^8 f(u) - u_0^8 f(u_0)}}. \quad (2.15)$$

The boundary conditions for the D8-brane profile are: asymptotically as $u \rightarrow \infty$, $x^4 \rightarrow L/2$ and $\partial_u x^4 \propto 1/u^{11/2} \rightarrow 0$; at the minimum $u \rightarrow u_0$, $x^4 \rightarrow 0^+$ and $\partial_u x^4 \propto 1/(u-u_0)^{1/2} \rightarrow \infty$. The full embedding consists of two halves of this form. Hence the D8- and $\overline{\text{D8}}$ -branes are joined in a smooth profile at the minimum radius u_0 and the two defects are separated by the asymptotic distance L in the x^4 direction. The limit $u_0 \rightarrow u_{\text{KK}}$ yields the trivial embedding $x^4 = \text{constant}$, which in the D4 soliton background corresponds to a smooth joining of

²See [21], for attempts to account for the gravitational back-reaction of the D8-branes.

the D8- and $\overline{D8}$ -branes with asymptotic separation $L = \pi r_4$. In the supersymmetric background ($u_{\text{KK}} = 0$) the trivial embedding is also allowed with an arbitrary separation L . In this case, the D8- and $\overline{D8}$ -branes terminate on the singularity at $u = 0$.

We can gain some intuition for these embeddings by considering the supersymmetric background. In this case with $u_{\text{KK}} = 0$, $f(u) = 1$ and hence the embedding equation reduces to

$$\frac{\partial x^4}{\partial u} = \frac{R^{3/2} u_0^4}{u^{3/2} \sqrt{u^8 - u_0^8}}. \quad (2.16)$$

Integrating gives a solution in terms of an incomplete Beta function [22]

$$x^4(u) = \frac{R^{3/2} \sqrt{\pi}}{8 \sqrt{u_0}} \frac{\Gamma(9/16)}{\Gamma(17/16)} - \frac{R^{3/2}}{8 \sqrt{u_0}} \beta \left(\frac{u_0^8}{u^8}, 9/16, 1/2 \right). \quad (2.17)$$

This has a finite $u \rightarrow \infty$ limit, so that the asymptotic (coordinate) separation of the $D8/\overline{D8}$ pair is given by

$$L = \lim_{u \rightarrow \infty} 2 x^4(u) = C_0 \frac{R^{3/2}}{u_0^{1/2}} \simeq 0.7245 \frac{R^{3/2}}{u_0^{1/2}}, \quad (2.18)$$

where $C_0 \equiv \frac{\sqrt{\pi} \Gamma(9/16)}{4 \Gamma(17/16)}$. Of course, L corresponds to the separation of the defects in the dual gauge theory. With the D4 soliton background there is a maximum separation corresponding to defects located at antipodes on the x^4 circle. In the supersymmetric background the x^4 coordinate need not be compact, so there is no restriction of this sort. However, other considerations bound the maximum value of L which one might consider in this case. Notice that increasing L corresponds to smaller values of u_0 . If u_0 becomes too small, the D8-brane extends into a region of high curvature and the calculation described above is no longer reliable. Therefore one can only reliably work with values of L where the minimum of the brane embedding is safely outside of this region.

With periodic fermions, the adjoint sector of the theory is supersymmetric and the gauge theory is not confining. In the dual background, free “constituent” quarks are realized as strings stretching from u_0 , the minimal radius of the branes, down to $u = 0$. The energy of these strings corresponds to the mass of the constituent quarks: $m_q = u_0/2\pi\alpha'$. With (2.18), this dynamically generated mass scale is given by

$$m_q = \frac{C_0^2}{8\pi^2} \frac{\lambda_5}{L^2} = \frac{C_0^2}{4\pi} g_{\text{YM}}^2 N_c \frac{r_4}{L^2}. \quad (2.19)$$

Since $2\pi r_4 \geq L$ we always have $m_q \gg 1/r_4$ in the strong coupling regime, which reflects the fact that the infrared dynamics does not decouple from the compactification scale. In the confining background of the D4 soliton background, there are no free quarks but one can still show that u_0 has the interpretation of roughly determining the constituent quark mass m_q as above [23], at least when u_0 is sufficiently larger than u_{KK} .

The constituent quarks above are complicated bound states of “current” quarks (i.e., the fundamental fields in the UV Lagrangian) and adjoint fields. This is shown in a striking way by comparing the quantum numbers of the constituent quarks to those of the current quarks. In particular, the current quarks are singlets under the global $\text{SO}(5)$ symmetry [3].

However, at u_0 , the D8-brane is wrapping the internal S^4 and the strings stretching from here to $u = 0$ in the supersymmetric background, can rotate in this internal space. Hence the constituent quarks transform nontrivially under $SO(5)$. Furthermore, it is likely that quantizing these strings will give a spectrum of both bosonic and fermionic states.

One can also look at the low-lying meson spectrum by considering excitations of the world-volume fields on the D8-branes [24, 25]. One finds that this spectrum is (as expected) characterized by the mass scale m_q/g_{eff} [26], in accord with the standard supergravity formula [18]. Here, one explicitly finds that mesons are both fermions and bosons [3], rather than just bosons. The latter reflects the fact that these infrared excitations are again complicated bound states of both the (fermionic) quarks and (both fermionic and bosonic) adjoint fields found in the UV Lagrangian.

3. Nonlocal mass term

Recall that the underlying microscopic field theory is a five-dimensional gauge theory with (chiral) fundamental matter fields localized on two four-dimensional defects. These defects are separated along the x^4 circle, so that fermions of different chiralities live at different places in the spacetime. Hence a naive mass term of the form $[\psi_L(x^\mu, x^4 = -L/2)]^a [\psi_R(x^\mu, x^4 = L/2)]_a$ is not possible — in particular, it is not gauge-invariant. As described in the introduction, the best one can do is to construct a nonlocal but gauge-invariant operator (1.1) with a Wilson line connecting the quarks on the separated defects [10–12]. This suggests that one consider the gravity/string dual as an instantonic Euclidean string worldsheet which sits at x^μ and extends between the D8-branes in the x^4 direction [13, 14, 27–29].³ Of course, this worldsheet does not quite reproduce the operator given above in (1.1). Rather this holographic construction introduces an “enhanced” Wilson line which sources both the gauge field and the adjoint scalars Φ^I of the five dimensional gauge theory [13]

$$\mathcal{O}(x^\mu) = [\psi_L(x^\mu, x^4 = 0)]^a P \exp \left[i \int_{-L/2}^{L/2} (A_4 - i n_I \Phi^I) dx^4 \right]_a^b [\psi_R(x^\mu, x^4 = L)]_b. \quad (3.1)$$

Here the (constant) normal vector n_I indicates the position of the worldsheet in the internal space. In principle, one could consider an elaborate contour for the Wilson line connecting the two fundamental fermions. However, for the sake of simplicity, we will only consider a straight contour (with fixed x^μ) in the following.

In analogy with the usual holographic calculations of Wilson loops, the expectation value of (3.1) is given by

$$\langle \mathcal{O} \rangle \sim e^{-I_{\text{ws}}}, \quad (3.2)$$

where I_{ws} is the worldsheet action for a string stretched between the $D8/\overline{D8}$ pair. However, there is an interesting difference between the present calculations and those for the conformal super-Yang-Mills theory [13, 14, 27, 28]. As described above, the five-dimensional

³A similar class of worldsheet instantons were studied with regard to the $U(1)_A$ problem in the Sakai-Sugimoto model [30].

gauge theory under consideration here is defined with a cut-off scale, $U_\infty = u_\infty/2\pi\alpha'$. The necessity of this cut-off is reflected in the nontrivial dilaton profile reaching strong coupling at a large radius in the dual D4-brane background. However, the standard Wilson line calculations are unaware of this aspect of the physics since those calculations only consider the leading-order Polyakov term for the worldsheet action in (3.2). The effect of the nontrivial dilaton is first seen at next-to-leading order in α' , through the Fradkin-Tseytlin term [31]. Hence to better understand the physics of Wilson line operators (3.1), we are motivated to carry the worldsheet calculations to first order in the α' expansion. Consistency at this order demands that we also include the fluctuation determinants of the worldsheet fields.

In the following, we find that the Fradkin-Tseytlin contribution generates new divergences in the calculation of (3.2) and hence the renormalization procedure for the worldsheet action must be modified. In particular, the new divergences do *not* seem to be removed by the Legendre transform introduced in [27]. We illustrate this in the following sections by evaluating the action for a specific worldsheet in the supersymmetric background. This simple calculation exhibits the full set of divergences that appear in subsequent calculations and so allows us to give a prescription for renormalizing the worldsheet action to first order in α' .

3.1 Rectangular worldsheet

To get a feeling for the issues that arise in the calculation of (3.2), we consider the trivial D8-brane embedding, $x^4 = \pm L/2$, in the supersymmetric D4-brane background. This corresponds to having the D8- and $\overline{\text{D8}}$ -branes extend straight down along the u direction, from the cut-off at u_∞ to the curvature singularity at $u = 0$. The gravity approximation breaks down for small values of u , so we introduce an (arbitrary) IR cut-off at $u = u_{\text{IR}}$, with $R^{3/2}u_{\text{IR}}^{1/2} \gg \alpha'$. For convenience, we assume that this is implemented by introducing a probe D4-brane where the strings can end. The result is the rectangular worldsheet shown in figure 1.

Let us begin by considering the Polyakov action

$$I_P = \frac{1}{4\pi\alpha'} \int_{\mathcal{M}} d^2\sigma \sqrt{g} g^{ab} G_{IJ} \partial_a X^I \partial_b X^J, \tag{3.3}$$

where \mathcal{M} is a worldsheet with boundary $\partial\mathcal{M}$, g_{ab} is the spacetime metric on \mathcal{M} , and G_{IJ} is the metric (2.2). Two of the worldsheet scalars can be identified with the coordinates u and x^4 — the remaining scalars can be ignored for the moment. The worldsheet metric is taken to be the same as the pullback of (2.2) to \mathcal{M} , which is given by

$$g_{ab} dx^a dx^b = \left(\frac{u}{R}\right)^{3/2} (dx^4)^2 + \left(\frac{R}{u}\right)^{3/2} du^2. \tag{3.4}$$

Evaluating the Polyakov action yields

$$I_{\text{P,rect}} = \frac{L}{2\pi\alpha'} (u_\infty - u_{\text{IR}}). \tag{3.5}$$

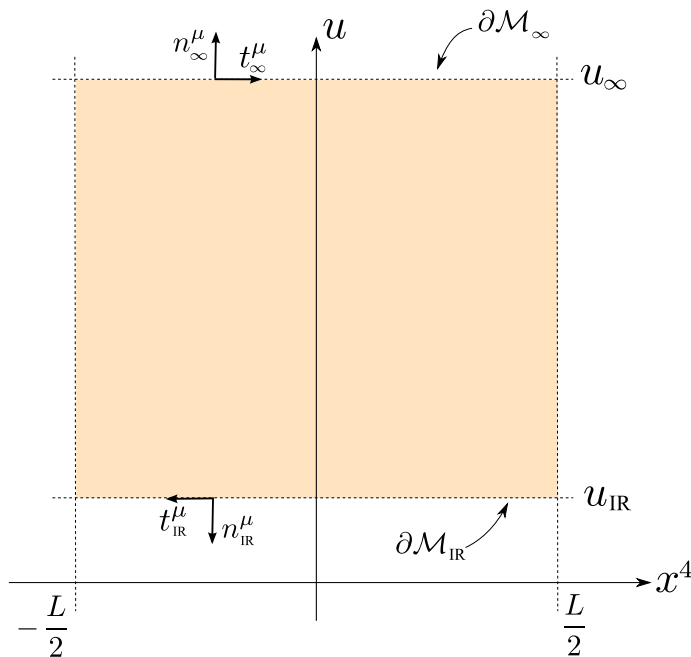


Figure 1: The rectangular worldsheet described in the main text.

The first contribution is a UV divergence proportional to the cut-off scale u_∞ . This term is removed by the Legendre transformation described in [27], which gives a ‘renormalized’ action

$$I_{\text{P,rect}}^{(\text{ren})} = -\frac{L}{2\pi\alpha'} u_{\text{IR}} . \tag{3.6}$$

Alternately, subtracting the UV-divergent term from (3.5) can be interpreted as a renormalization of the field theory operator (3.1).

The Polyakov action is the leading order contribution to the worldsheet action in the α' expansion. We must also take into account terms at the next order in this expansion that couple to the nontrivial dilaton of the D4-brane background. Specifically, we have to evaluate the Fradkin-Tseytlin term [31, 32]

$$I_{\text{FT}} = \frac{1}{4\pi} \int_{\mathcal{M}} d^2\sigma \sqrt{g} \mathcal{R}(g) \Phi + \frac{1}{2\pi} \int_{\partial\mathcal{M}} ds \mathcal{K} \Phi + \frac{1}{2\pi} \sum_i \Phi(x_i^\mu) (\pi - \theta_i) , \tag{3.7}$$

where \mathcal{R} is the worldsheet Ricci scalar, s is the proper distance along the boundary $\partial\mathcal{M}$, and \mathcal{K} is the geodesic curvature of the boundary. The latter is defined as

$$\mathcal{K} = -t^a n_b \nabla_a t^b \tag{3.8}$$

where t^a and n^a are unit vectors tangent and normal to the boundary, respectively. The last term in (3.7) is a sum over corners where the embedding of the boundary is not smooth. A corner that makes an angle θ gives a contribution proportional to $\pi - \theta$, times the value of the dilaton at that point. These corner terms can be thought of as arising from δ -function contributions to the geodesic curvature. Of course, with a constant dilaton Φ_0

the sum of all the terms in (3.7) gives $I_{\text{FT}} = \chi \Phi_0$, where χ is the Euler character of the worldsheet. The worldsheets that we consider all have the topology of a disk, i.e., $\chi = 1$. Finally, consistency requires that we take into account the fluctuation determinant on the worldsheet at this order in α' . This calculation is performed in appendix A, where we find that it does not make a significant contribution to the worldsheet action. In particular, these one-loop determinants do not generate any additional UV divergences.

We now evaluate the individual terms in (3.7), beginning with the scalar curvature term. The Ricci scalar for the worldsheet metric (3.4) is given by

$$\mathcal{R}(g) = -\frac{3}{4} \frac{1}{R^{3/2} u^{1/2}}. \quad (3.9)$$

and the first term in (3.7) is

$$\begin{aligned} I_{\mathcal{R}} &= \frac{1}{4\pi} \int_{-L/2}^{L/2} dx^4 \int_{u_{\text{IR}}}^{u_{\infty}} du \left(-\frac{3}{4} \frac{1}{R^{3/2} u^{1/2}} \log \left(g_s (u/R)^{3/4} \right) \right) \\ &= \frac{3L}{8\pi R} \left[\left(\frac{u}{R} \right)^{1/2} \left(\frac{3}{2} - \log \left(g_s (u/R)^{3/4} \right) \right) \right]_{u_{\text{IR}}}^{u_{\infty}}. \end{aligned}$$

Next we consider the contributions to (3.7) from the smooth components of the boundary. The component of the boundary extending from $(-L/2, u_{\infty})$ to $(L/2, u_{\infty})$ has tangent and normal vectors given by

$$t^a = (R/u_{\infty})^{3/4} (\partial/\partial x^4)^a, \quad n^a = (u_{\infty}/R)^{3/4} (\partial/\partial u_{\infty})^a. \quad (3.10)$$

Using these expressions in equation (3.8) gives the geodesic curvature along this part of the boundary

$$\mathcal{K} = \frac{3}{4R} \left(\frac{R}{u_{\infty}} \right)^{1/4}. \quad (3.11)$$

The proper distance along this edge is $ds = (u_{\infty}/R)^{3/4} dx^4$, so the contribution to the action is

$$I_{\mathcal{K}} = \frac{1}{2\pi} \int_{-L/2}^{L/2} ds \mathcal{K} \Phi = \frac{3L}{8\pi R} \left(\frac{u_{\infty}}{R} \right)^{1/2} \log \left[g_s \left(\frac{u_{\infty}}{R} \right)^{3/4} \right]. \quad (3.12)$$

The component of the boundary between $(-L/2, u_0)$ to $(L/2, u_0)$ makes a similar contribution; it differs by an overall minus sign and the substitution $u_{\infty} \rightarrow u_0$. The geodesic curvature vanishes for the edges of the worldsheet along $x^4 = \pm L/2$, so they do not contribute to the action. Finally we consider the contribution of the four corners of the worldsheet, each of which makes an angle $\theta_i = \pi/2$. Their contribution to the action is

$$I_{\text{corners}} = \frac{1}{2} \log \left[g_s \left(\frac{u_{\text{IR}}}{R} \right)^{3/4} \right] + \frac{1}{2} \log \left[g_s \left(\frac{u_{\infty}}{R} \right)^{3/4} \right]. \quad (3.13)$$

Collecting these terms, the action (3.7) yields

$$I_{\text{FT,rect}} = \frac{9L}{16\pi R} \left(\sqrt{\frac{u_{\infty}}{R}} - \sqrt{\frac{u_{\text{IR}}}{R}} \right) + \frac{3}{8} \log \left[\frac{u_{\infty} u_{\text{IR}}}{R^2} \right] + \log g_s. \quad (3.14)$$

Hence the inclusion of (3.7) leads to two new UV-divergent terms in the worldsheet action, proportional to $\sqrt{u_\infty}$ and $\log u_\infty$. These are in addition to the divergent term coming from the Polyakov action (3.5). Notice as well that both (3.10) and (3.12) contained potentially divergent terms of the form $u_\infty^{1/2} \log u_\infty$, however, these terms cancel out in the final expression.

Although we have used a particularly simple background and worldsheet configuration in the present calculation, the structure of the UV divergences depends only on the asymptotic behaviour. This means that the result obtained here is in fact universal, and the divergences we have found also appear in more general situations. The calculations in section 4 — both analytical and numerical — show this explicitly. Therefore, as we discuss below, the results of this section lead to a general prescription for renormalizing the worldsheet action and obtaining a finite expectation value $\langle \mathcal{O} \rangle$.

As a final comment here, we note the term $\log g_s$ which arises in (3.14) from the inclusion of the Fradkin-Tseytlin term (3.7) in our calculation. Keeping the background scale R fixed, (2.6) gives $g_s \propto 1/N_c$ and hence one finds $\langle \mathcal{O} \rangle \sim N_c$. Given that \mathcal{O} is a bilinear of fields in the fundamental representation of the gauge group, this latter factor is precisely the expected result by the standard large N_c counting. Of course, this factor is a universal result for all such worldsheet calculations, as we will see with the examples calculated in the section 4.

3.2 Renormalization of the worldsheet action

One can try to address the UV divergences in (3.14) by applying the Legendre transform described in [27]. The authors there demonstrated that the ‘correct’ action for observables related to the minimal area of a string worldsheet is the Legendre transform of (3.3) with respect to some of the loop variables — see also [28]. This is because some of the worldsheet scalars satisfy Neumann boundary conditions asymptotically rather than Dirichlet boundary conditions. Indeed, as we commented above, implementing this Legendre transformation removes the UV-divergent term from the Polyakov action (3.5). However, a straightforward application of the same Legendre transform does not cancel the divergent terms in (3.14).

To see that this is the case, first vary the full worldsheet action with respect to the worldsheet fields. This gives an expression of the form

$$\delta I = \int_{\mathcal{M}} d^2x \sqrt{g} \left[\mathcal{E}^{\mu\nu} \delta g_{\mu\nu} + \mathcal{E}_I \delta X^I \right] + \int_{\partial\mathcal{M}} dx \sqrt{h} \left[\pi^{\mu\nu} \delta h_{\mu\nu} + \pi_I \delta X^I \right] \quad (3.15)$$

The coefficients of $\delta g_{\mu\nu}$ and δX^I in the integral over \mathcal{M} are the worldsheet equations of motion, while the coefficients of $\delta h_{\mu\nu}$ and δX^I in the boundary integral are the momenta pulled back to $\partial\mathcal{M}$. The π_I are given by

$$\pi_I = \frac{1}{2\pi\alpha'} G_{IJ} n^\mu \partial_\mu X^J + \frac{1}{2\pi} \mathcal{K} \partial_I \Phi \quad (3.16)$$

where n^μ is an outward pointing unit vector normal to $\partial\mathcal{M}$, and all fields are evaluated at $\partial\mathcal{M}$. The Legendre transform of the action with respect to some subset $\{X^J\}$ of the

worldsheet scalars is denoted \tilde{I} and is given by

$$\tilde{I} = I - \int_{\partial\mathcal{M}} dx \sqrt{h} \sum_{\{X^J\}} \pi_J X^J . \quad (3.17)$$

Following [27], we construct the Legendre transform of I_{rect} with respect to the worldsheet scalar u at u_∞

$$\tilde{I}_{\text{rect}} = I_{\text{rect}} - \int_{\partial\mathcal{M}_\infty} dx \sqrt{h} \pi_u u . \quad (3.18)$$

Using (3.16), we have

$$\pi_u = \frac{1}{2\pi\alpha'} \left(\frac{R}{u}\right)^{3/4} + \frac{9}{32\pi} \frac{1}{R^{3/4}u^{5/4}} \quad (3.19)$$

The induced metric at u_∞ yields $\sqrt{h} = (u_\infty/R)^{3/4}$ and so

$$\int_{\partial\mathcal{M}_\infty} dx \sqrt{h} \pi_u u = \frac{L}{2\pi\alpha'} u_\infty + \frac{9L}{32\pi R^{3/4}} \sqrt{u_\infty} . \quad (3.20)$$

The first term cancels the leading power-law divergence coming from the Polyakov term (3.5), but the second term does not cancel the corresponding $u_\infty^{1/2}$ term in (3.14)

$$\tilde{I}_{\text{rect}} = -\frac{L}{2\pi\alpha'} u_0 + \frac{9L}{32\pi R^{3/2}} (\sqrt{u_\infty} - 2\sqrt{u_0}) + \frac{1}{2} (\Phi_\infty + \Phi_0) . \quad (3.21)$$

Thus the usual Legendre transform of the worldsheet action does not address the divergent terms at next-to-leading order in α' . As the structure of the UV divergences is universal, this approach also fails for more general curved embeddings, such as those that we study in the next section.

The simplest method for dealing with the divergences is to subtract the terms in (3.14) that depend on u_∞ . This approach is closer in spirit to that applied in the holographic renormalization of probe D-brane calculations, e.g., see [33, 34]. We have already noted that subtracting the u_∞ term in the Polyakov action can be interpreted in the field theory as a UV renormalization of \mathcal{O} and the same interpretation applies to the new terms at next-to-leading order in the worldsheet action. Such a subtraction is straightforward for the $u_\infty^{1/2}$ term, however, we also have to deal with the logarithmic contribution from the two corners at $u = u_\infty$, $x^4 = \pm L/2$. An ambiguity naturally arises here because the subtraction, which takes the form $3/4 \log(u_\infty/u_{\text{sub}})$, requires the introduction of a subtraction scale u_{sub} . Thus, our proposal for renormalizing the worldsheet action is

$$I_{\text{WS}}^{(\text{ren})} = I_{\text{WS}} - \frac{L}{2\pi\alpha'} u_\infty - \frac{9L}{16\pi R} \sqrt{\frac{u_\infty}{R}} - \frac{3}{8} \log\left(\frac{u_\infty}{u_{\text{sub}}}\right) . \quad (3.22)$$

The UV divergences that we subtract are universal and render the worldsheet action finite up to terms of order α' . As shown in appendix A, there are no divergences associated with the fluctuation determinant. In the next section we apply this renormalization to the worldsheet action for a string stretching between the $D8$ - and $\overline{D8}$ -branes with the curved embedding.

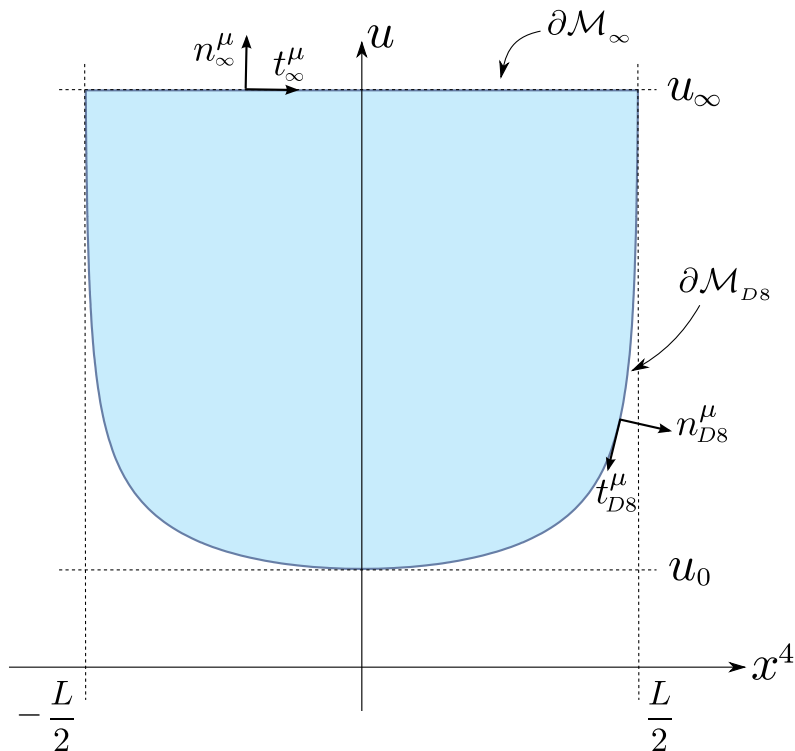


Figure 2: Worldsheet for a smooth embedding of the $D8/\overline{D8}$ branes.

4. Worldsheet for smooth $D8-\overline{D8}$ pair embedding

Now we turn to an explicit calculation of the expectation value $\langle \mathcal{O} \rangle$ for the Sakai-Sugimoto model [3, 4]. As this holographic model does not permit the construction of a local fermion bilinear, this expectation value is the condensate which characterizes the spontaneous breaking of chiral symmetry in this model. As described in section 2, this spontaneous symmetry breaking is realized in the gravitational dual by the $D8-\overline{D8}$ pair joining together to form a smooth U-shaped embedding, as illustrated in figure 2. In the figure, the light blue region is the (Euclidean) worldsheet of a string stretching between the $D8-\overline{D8}$ pair. The boundary of this worldsheet consists of two smooth components: the segment $\partial\mathcal{M}_\infty$ defined by the cut-off u_∞ , and the segment $\partial\mathcal{M}_{D8}$ defined by the embedding $x^4 = x^4(u)$.

Recall that embedding equation (2.15) determines the D8-brane profile $x^4(u)$

$$\frac{\partial x^4}{\partial u} = \frac{R^{3/2} u_0^4 \sqrt{f(u_0)}}{u^{3/2} f(u)} \frac{1}{\sqrt{u^8 f(u) - u_0^8 f(u_0)}}. \quad (4.1)$$

where u_0 is the minimum radius where the $D8-\overline{D8}$ pair joins smoothly — see figure 2 — and $f(u) = 1 - (u_{\text{KK}}/u)^3$. With $u_{\text{KK}} \neq 0$, this equation cannot be solved analytically and so the embedding must be determined numerically.

To numerically solve for the embedding we define the following dimensionless variables

$$z = \frac{u}{u_0} \quad \omega = \frac{u_{\text{KK}}}{u_0} \quad \psi = \sqrt{\frac{u_0}{R}} \frac{x^4}{R}, \quad (4.2)$$

with u_0 the point on the u -axis where the embedding reaches its minimum value. The restriction $u_0 \leq u \leq u_\infty$ implies $1 \leq z \leq z_\infty$. Similarly, the parameter ω takes values $\omega \in [0, 1]$. Here the lower limit corresponds to $u_0 \rightarrow \infty$ but this limit is also realized in the extremal background with $u_{\text{KK}} = 0$. The upper bound is reached when the embedding reaches the minimum radius of the background at $u = u_{\text{KK}}$. In terms of these dimensionless variables, the embedding equation becomes

$$\frac{\partial\psi(z, \omega)}{\partial z} = \frac{z^{3/2} \sqrt{1 - \omega^3}}{z^3 - \omega^3} \frac{1}{\sqrt{z^8 - \omega^3 z^5 - 1 + \omega^3}}. \quad (4.3)$$

Solving this equation numerically yields a family of embeddings $\psi(z, \omega)$ parameterized by ω .

With the standard dictionary $U = u/2\pi\ell_s^2$, the parameter ω becomes a ratio of scales $\omega = \frac{U_{\text{KK}}}{U_0}$ where U_{KK} and U_0 can be thought of as the confinement and chiral symmetry breaking scales, respectively. In this model, these are both dynamically generated scales determined by the fundamental gauge theory parameters. For example, using (2.5) and the subsequent formulae in section 2.1, $U_{\text{KK}} = \frac{2}{9} \frac{\lambda_5}{(2\pi r_4)^2}$ and in the supersymmetric background, $U_0 = m_q$ in (2.19). Hence, in principle ω is also a function of the parameters L , r_4 and λ_5 . In fact, a relatively simple expression can be derived by first noting that for a sufficiently large cut-off $z_\infty = u_\infty/u_0$, $\psi(z_\infty, \omega)$ is essentially only a function of ω . Then the asymptotic separation $L = 2x^4(u_\infty)$ can be expressed in terms of $\psi_\infty(\omega)$ with

$$L = 2\psi_\infty(\omega) \sqrt{\frac{R^3}{u_0}}. \quad (4.4)$$

Using the various expressions in section 2.1, we then find

$$\frac{1}{3} \frac{L}{r_4} = \sqrt{\omega} \psi_\infty(\omega) \quad (4.5)$$

and so in fact ω is independent of the coupling λ_5 . Hence the coupling dependence of the individual scales U_{KK} and U_0 has canceled in the ratio defining ω . The function $\sqrt{\omega} \psi_\infty(\omega)$ is shown in figure 3 on the range $\omega \in [0, 1]$.

Holding L fixed in (4.5) implicitly gives ω as a function of r_4 . This allows us to interpret $\psi(z, \omega)$ as a family of $D8/\overline{D8}$ embeddings with constant asymptotic separation in $D4$ -brane backgrounds with different compactification radii r_4 . Three such embeddings are shown in figure 4. Alternately, fixing r_4 in (4.5) implicitly gives ω as a function of L . In that case $\psi(z, \omega)$ represents a family of $D8/\overline{D8}$ embeddings with varying asymptotic separation in a fixed, non-extremal $D4$ -brane background.

4.1 Worldsheet action for the curved $D8/\overline{D8}$ embedding

Next we explicitly evaluate the worldsheet action for a string stretched between $D8$ - and $\overline{D8}$ -branes for the curved embeddings described in the previous section. In the extremal background the calculation can be performed analytically; in the non-extremal case we must use numerical methods. While its contributions are subdominant, we include the Fradkin-Tseytlin term in the following for illustrative purposes.

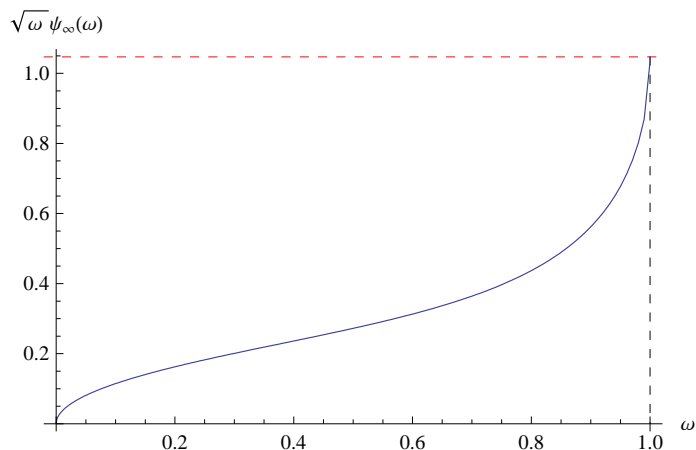


Figure 3: The function $\sqrt{\omega} \psi_\infty(\omega)$ that appears on the right-hand-side of equation (4.5). The dashed red line corresponds to $L = \pi r_4$, where the defects sit at antipodal points on the x^4 circle. The dashed black line at $\omega = 1$ is the bound $u = u_{\text{KK}}$.

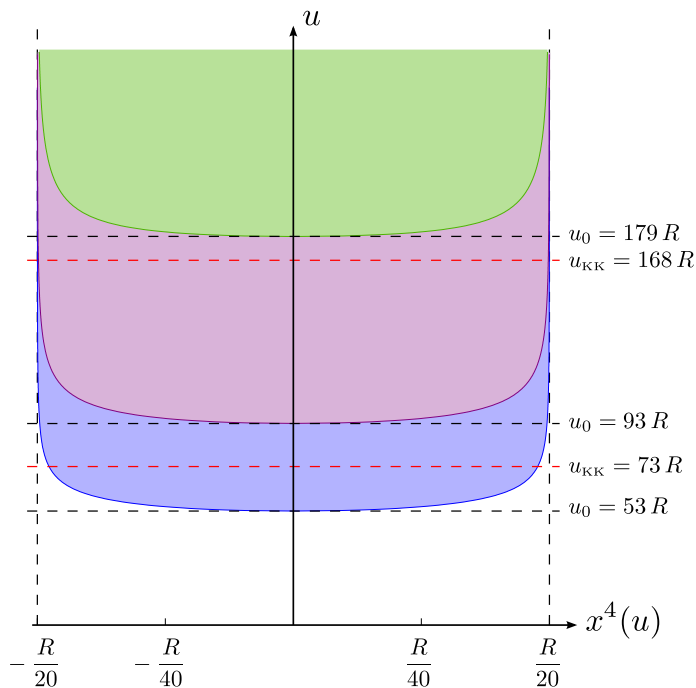


Figure 4: $D8/\overline{D8}$ embeddings with fixed asymptotic separation $L = R/10$, for $u_{\text{KK}} = 0$ (blue), $u_{\text{KK}} = 73 R$ (violet), and $u_{\text{KK}} = 168 R$ (green). The non-zero values of u_{KK} are indicated by red dashed lines, and the minimum value of u reached by each embedding is indicated by a black dashed line.

As in the previous calculation, we identify the coordinates of the Euclidean worldsheet with the spacetime coordinates u and x^4 . The worldsheet metric and dilaton are given by

$$ds^2 = \left(\frac{u}{R}\right)^{3/2} f(u) (dx^4)^2 + \left(\frac{R}{u}\right)^{3/2} \frac{du^2}{f(u)} \quad \Phi = \log \left(g_s \left(\frac{u}{R}\right)^{3/4} \right). \quad (4.6)$$

To compute the Fradkin-Tseytlin part of the action we need expressions for the scalar curvature on \mathcal{M} , as well as the proper distance and geodesic curvature on $\partial\mathcal{M}$. The Ricci scalar for the metric (4.6) is

$$\mathcal{R} = -\frac{3}{4R^{3/2}u^{1/2}} + \frac{15u_{\text{KK}}^3}{4R^{3/2}u^{7/2}}. \quad (4.7)$$

The normal vector, tangent vector, and geodesic curvature for the component of the boundary at $u = u_\infty$ are given by

$$n_\mu = \left(\frac{u_\infty}{R}\right)^{3/4} \frac{1}{\sqrt{f(u_\infty)}} \delta_\mu^u \quad t_\mu = \left(\frac{R}{u_\infty}\right)^{3/4} \sqrt{f(u_\infty)} \delta_\mu^{x^4} \quad (4.8)$$

$$\mathcal{K} = \frac{3(u_\infty^3 + u_{\text{KK}}^3)}{4R^{3/4}u_\infty^{13/4}\sqrt{f(u_\infty)}}. \quad (4.9)$$

On the component of $\partial\mathcal{M}$ described by the embedding $x^4(u)$ these quantities are

$$n_{x^4} = \pm \left(\frac{u}{R}\right)^{3/4} \frac{\sqrt{u^8 f(u) - u_0^8 f(u_0)}}{u^4} \quad n_u = - \left(\frac{R}{u}\right)^{3/4} \left(\frac{u_0}{u}\right)^4 \frac{\sqrt{f(u_0)}}{f(u)} \quad (4.10)$$

$$t_{x^4} = - \left(\frac{u}{R}\right)^{3/4} \left(\frac{u_0}{u}\right)^4 \sqrt{f(u_0)} \quad t_u = \mp \left(\frac{R}{u}\right)^{3/4} \frac{\sqrt{u^8 f(u) - u_0^8 f(u_0)}}{u^4 f(u)} \quad (4.11)$$

$$\mathcal{K} = \frac{13u_0^4 \sqrt{f(u_0)}}{4R^{3/4}u^{17/4}}. \quad (4.12)$$

Above in (4.10) and (4.11), the upper (lower) sign corresponds to the portion of the boundary with $x^4 > 0$ ($x^4 < 0$). Using these expressions and the dimensionless variables (4.2), the worldsheet action is

$$\begin{aligned} I_{\text{WS}} &= \frac{R^2}{\ell_s^2} \sqrt{\frac{u_0}{R}} \frac{1}{\pi} \int_1^{z_\infty} dz \psi(z, \omega) + \frac{3}{8\pi} \int_1^{z_\infty} dz \psi(z, \omega) \left(-\frac{3}{4z^{1/2}} + \frac{15\omega^3}{4z^{7/2}} \right) \log z \\ &+ \frac{9}{16\pi} \psi_\infty(\omega) z_\infty^{1/2} \left(1 + \frac{\omega^3}{z_\infty^3} \right) \log z_\infty + \frac{39}{16\pi} \int_1^{z_\infty} dz \frac{\sqrt{1-\omega^3}}{z \sqrt{z^8 - \omega^3 z^5 - 1 + \omega^3}} \log z \\ &+ \frac{3}{8} \log z_\infty + \log \left(g_s \left(\frac{u_0}{R} \right)^{3/4} \right). \end{aligned} \quad (4.13)$$

This expression must be renormalized according to the prescription in section 3.2, which in terms of the dimensionless variables becomes

$$I_{\text{WS}}^{(\text{ren})} = I_{\text{WS}} - \frac{L u_0}{2\pi \alpha'} z_\infty - \frac{9L \sqrt{u_0}}{16\pi R^{3/2}} \sqrt{z_\infty} - \frac{3}{8} \log(z_\infty/z_{\text{sub}}). \quad (4.14)$$

As described above, this prescription requires choosing a subtraction scale $u_{\text{sub}} = u_0 z_{\text{sub}}$. For simplicity, we choose $u_{\text{sub}} = u_0$ (i.e., $z_{\text{sub}} = 1$) in the following. To proceed, we must use (4.4) to simplify various factors, e.g.,

$$\frac{L u_0}{2\pi \alpha'} = \frac{\lambda_5}{2\pi L} \frac{\psi_\infty^2(\omega)}{\pi}, \quad \frac{R^2}{\ell_s^2} \sqrt{\frac{u_0}{R}} = \frac{\lambda_5}{2\pi L} \psi_\infty(\omega). \quad (4.15)$$

In particular, with these expressions, the final result is expressed as a function of the ratio λ_5/L and the parameter ω . Then using (4.14) to explicitly remove the divergent terms from (4.13), the renormalized action becomes

$$I_{\text{WS}}^{(\text{ren})} = -\frac{\lambda_5}{2\pi L} \psi_\infty(\omega) F_1(\omega) - F_2(\omega) - \log(\pi N_c) + \frac{3}{2} \log\left(\frac{\psi_\infty(\omega)}{2\pi} \frac{\lambda_5}{L}\right). \quad (4.16)$$

where

$$F_1(\omega) = \frac{1}{\pi} \int_1^{z_\infty} dz \left(\psi_\infty(\omega) - \psi(z, \omega) \right) + \frac{1}{\pi} \psi_\infty(\omega) \quad (4.17)$$

$$\begin{aligned} F_2(\omega) = & -\frac{9}{32\pi} \int_1^{z_\infty} dz \left(\psi_\infty(\omega) - \psi(z, \omega) \right) \frac{\log z}{\sqrt{z}} + \frac{9}{8\pi} \psi_\infty(\omega) \quad (4.18) \\ & -\frac{9\omega^3}{16\pi} \psi_\infty(\omega) \frac{\log z_\infty}{z_\infty^{5/2}} - \frac{45\omega^3}{32\pi} \int_1^{z_\infty} dz \psi(z, \omega) \frac{\log z}{z^{7/2}} \\ & -\frac{39}{16\pi} \int_1^{z_\infty} dz \frac{\sqrt{1-\omega^3}}{z \sqrt{z^8 - z^5 \omega^3 - 1 + \omega^3}} \log z. \end{aligned}$$

The function $F_1(\omega)$ is strictly positive, while $F_2(\omega)$ is bounded from below. Further we note that the renormalization indicated in (4.14) has been incorporated in the definitions of these functions in such a way each of the individual integrals appearing in (4.17) and (4.18) is manifestly finite.

The functions $F_1(\omega)$ and $F_2(\omega)$ appearing in the action (4.16) are obtained in general by numerically performing the integrals in (4.17) and (4.18). However, in the extremal $D4$ -brane background (with $u_{\text{KK}} = 0$) an analytical expression can be given for the renormalized worldsheet action (4.16). Using the embedding (2.17) of the $D8$ - $\overline{D8}$ pair in the extremal background, the renormalized worldsheet action is

$$I_{\text{WS}}^{(\text{ren})}(\omega = 0) = -\frac{1}{8\pi} \tan\left(\frac{\pi}{16}\right) \cdot \frac{\lambda_5}{L} - \frac{3}{2} \log\left(\frac{L}{\lambda_5}\right) - \log N_c - \frac{9}{64} + \frac{3}{2} \log\left(\frac{2^{1/16} \Gamma(9/16)}{\pi^{7/6} \Gamma(1/16)}\right), \quad (4.19)$$

in the limit that $z_\infty \rightarrow \infty$. Thus, the expectation value of $\langle \mathcal{O} \rangle$ takes the form

$$\langle \mathcal{O} \rangle \sim N_c \left(\frac{L}{\lambda_5}\right)^{3/2} \exp\left(\frac{1}{8\pi} \tan\left(\frac{\pi}{16}\right) \frac{\lambda_5}{L}\right). \quad (4.20)$$

The exponential dependence on λ_5/L is precisely that found in [10] coming from the Polyakov action. As described above, the overall factor of N_c comes from the Fradkin-Tseytlin contribution (3.7) to the action. This term also produces the pre-factor of $(L/\lambda_5)^{3/2}$. However, one should keep in mind that a complete calculation at this order in α' expansion would require evaluating the fluctuation determinant on the string worldsheet. Hence one should expect this pre-factor to be modified in a complete evaluation at this order.

Given the general result for the renormalized action (4.16), the expectation value of the operator \mathcal{O} is given by

$$\langle \mathcal{O} \rangle \sim N_c \left(\frac{L}{\lambda_5}\right)^{3/2} \psi_\infty(\omega)^{-3/2} \exp\left(\frac{\lambda_5}{L} \frac{\psi_\infty(\omega)}{2\pi} F_1(\omega) + F_2(\omega)\right). \quad (4.21)$$

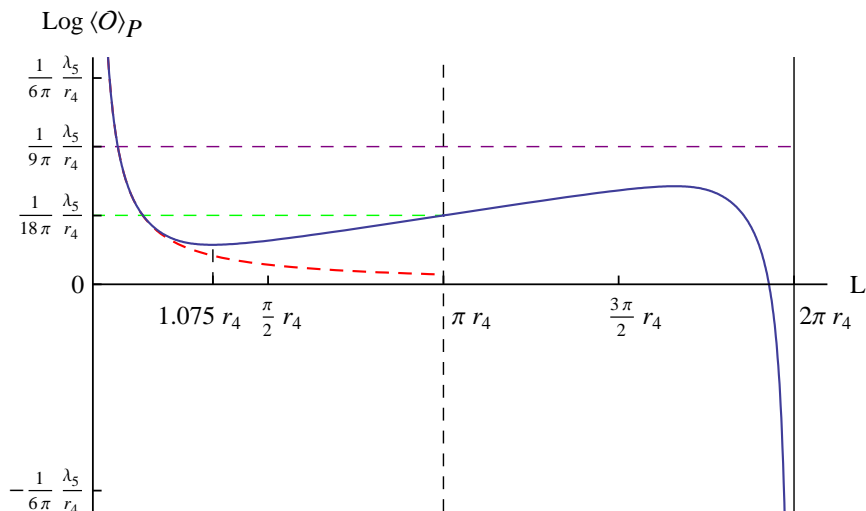


Figure 5: The contribution to $\log\langle\mathcal{O}\rangle$ from the renormalized Polyakov action as a function of L , with r_4 and λ_5 fixed. The dashed red curve is the result for the extremal background. The dashed vertical line at $L = \pi r_4$ corresponds to the defects positioned at antipodal points on the x^4 circle. The value of $\log\langle\mathcal{O}\rangle$ for this configuration is indicated by the dashed green line. The dashed purple line shows the value for the closed Wilson line $\langle\mathcal{W}\rangle$.

where implicitly we have again taken the limit $z_\infty \rightarrow \infty$. The functions $F_1(\omega)$ and $F_2(\omega)$ must be determined numerically in the nonextremal background with $u_{\text{KK}} \neq 0$. As a check of our numerical calculations, the action for the $\omega = 0$ case was determined numerically and compared with (4.19). In addition, the Euler number was calculated numerically for each $\omega \neq 0$ embedding and compared with the expected value: $\chi = 1$. In both cases the numerical error, expressed as a fraction of the expected result, was of order of 10^{-10} to 10^{-11} .

As before, the Polyakov action (i.e., the F_1 term in (4.21)) dominates in the supergravity limit and so we focus on this term in the following. The result depends on all three of the independent parameters, λ_5 , r_4 and L (or rather dimensionless ratios of these parameters) — recall that ω is implicitly defined as a function of the ratio $\frac{L}{r_4}$ by the relation (4.5). A natural approach is to hold the gauge theory parameters constant (by fixing the ratio λ_5/r_4) and consider the expectation value as a function of L , the separation of the defects. This is illustrated in figure 5, where we show $\log\langle\mathcal{O}\rangle \sim \frac{\lambda_5}{L} \frac{\psi_\infty(\omega)}{2\pi} F_1(\omega)$ as a function of L . The plot shows that as L/r_4 approaches zero, our result follows the extremal result, $\tan(\pi/16) \frac{\lambda_5}{8\pi L}$, appearing in the exponential factor in (4.20). This behaviour arises because for $L \ll r_4$, the $D8/\overline{D8}$ branes do not extend very far into the bulk and so the string stretched between them detects no difference between the extremal and non-extremal backgrounds. In terms of the gauge theory, this behaviour simply reflects the fact that the chiral symmetry breaking scale U_0 is much larger than the confining scale U_{KK} , where supersymmetry breaking takes effect in the gauge theory. As L becomes larger, the calculation begins to probe regions of the dual spacetime geometry closer to $u = u_{\text{KK}}$ and one sees that the extremal and nonextremal behaviours of $\log\langle\mathcal{O}\rangle$ begin to deviate around $L \simeq r_4$. The expectation value reaches an interesting local minimum at $L \sim 1.075 r_4$, where $U_0/U_{\text{KK}} \simeq 1.608$. Note

that the location of the minimum is independent of λ_5/r_4 and is therefore always visible in the supergravity limit.

When L reaches πr_4 , the defects are located at antipodal points on the x^4 circle. This corresponds, in the relation (4.5), to the limiting value $\omega = 1$. However, we have extended L to the region $\pi r_4 \leq L \leq 2\pi r_4$ in figure 5. Of course, in this regime, the shortest distance between the defects on the x^4 circle is $\tilde{L} = 2\pi r_4 - L$ but the open Wilson line stretches the longer distance L around the circle.⁴ The embedding profile of the D8- $\overline{\text{D8}}$ pair is identical to those in section 4.1 but with \tilde{L} replacing L . Now in the expectation value, the dual worldsheet spans the minimal surface ‘outside’ of the U-shape formed by the D8- $\overline{\text{D8}}$ pair. Hence the (renormalized) Polyakov action may be calculated as the action of a worldsheet covering the entire u - x^4 geometry (4.6) minus that for the worldsheet stretched ‘inside’ of the U-shape. As a result, at this order, we have the relation: $\langle \mathcal{O} \rangle(\tilde{L}/r_4 = 2\pi - L/r_4) \times \langle \mathcal{O} \rangle(L/r_4) = \langle \mathcal{W} \rangle$ where $\langle \mathcal{W} \rangle$ is the expectation value of a closed Wilson line which winds once around the x^4 circle. (Note that we find $\log \langle \mathcal{W} \rangle \simeq \lambda_5/(9\pi r_4)$.) Figure 5 displays a symmetry about $L = \pi r_4$ which reflects this relation and we may infer that $\langle \mathcal{O} \rangle(L/r_4)$ approaches zero as $L \rightarrow 2\pi r_4$. Of course, we should add that the five-dimensional gauge theory is defined with a cut-off U_∞ and so one should not really consider the above results for $L, \tilde{L} \lesssim 1/U_\infty$.

An alternative approach to considering the expectation value $\langle \mathcal{O} \rangle$, is to consider it as a function of r_4 with fixed L and λ_5 . The result is plotted in figure 6 for $r_4 > L/\pi$. In the decompactification limit, $r_4 \rightarrow \infty$, the expectation value again asymptotes to the extremal result $\tan(\pi/16) \frac{\lambda_5}{8\pi L}$. Note that, because ω is independent of the ratio λ_5/L , this plot can also be understood as showing the dependence of $\langle \mathcal{O} \rangle$ on the four-dimensional ‘t Hooft coupling, using the relation $\lambda_4 = \lambda_5/(2\pi r_4)$.

5. Discussion

We have examined various aspects of a recent proposal [10–12] to add quark masses to the Sakai-Sugimoto model with nonlocal operators of the form (3.1). The underlying microscopic field theory is a five-dimensional gauge theory where the chiral quarks are localized on separate four-dimensional defects. However, the five-dimensional gauge theory is only defined with a cut-off, i.e., new degrees of freedom appear in the far UV. In the dual supergravity background, this issue is realized by the running of the dilaton which produces large string coupling in the asymptotic region. In section 3, we examined modifications introduced by the coupling of the dilaton to the string worldsheet. In particular, we showed that this coupling calls for a modification of the renormalization of these operators as in (3.22). The first two subtractions, which are linear in the length L , renormalize the Wilson line and are not particular to the present open Wilson line calculations. Both of these terms, including the second one proportional to $\sqrt{u_\infty}$, would also appear in calculations for closed Wilson lines. From the point of view of the five-dimensional gauge theory, these subtractions implement a renormalization scheme in which a straight, infinitely long Wilson line

⁴In principle, the following construction could be extended to consider Wilson lines which connect the defects after fully winding around the x^4 circle some number of times.

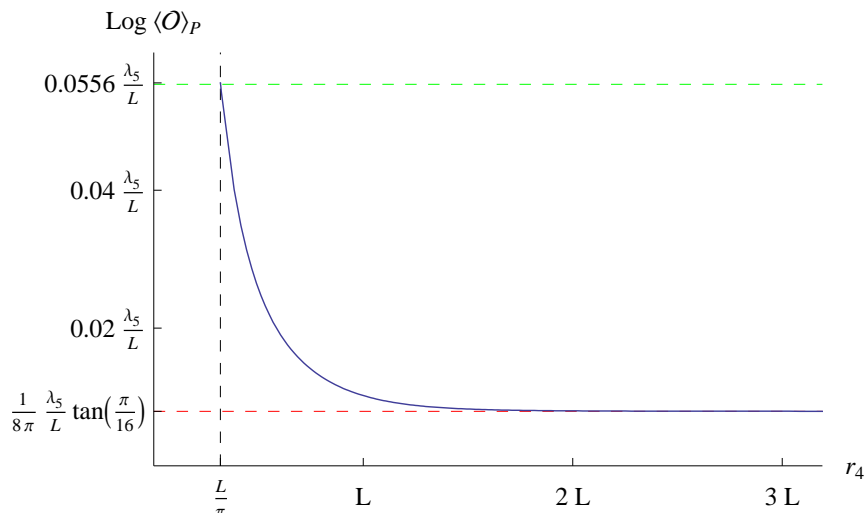


Figure 6: The contribution to $\log\langle\mathcal{O}\rangle$ from the Polyakov action as a function of r_4 , with L and λ_5 fixed. For large r_4 the value approaches the result for the extremal background, shown as a dashed red line.

would have expectation value $\langle\mathcal{O}\rangle = 1$. On the other hand, the log subtraction is distinctive of the two end-points of the open Wilson line and introduces a certain ambiguity, as described above in section 3.2. Addressing the log divergence requires a choice of subtraction scale u_{sub} and the choice made in our calculations was simply guided by convenience.

It is interesting to re-express the subtractions in (3.22) in terms of an energy cut-off, using the standard dictionary $U_\infty = u_\infty/2\pi\ell_s^2$,

$$I_{\text{WS}}^{(\text{ren})} = I_{\text{WS}} - L U_\infty - \frac{9}{4\sqrt{2}} \frac{L U_\infty}{g_{\text{eff}}(U_\infty)} - \frac{3}{8} \log\left(\frac{U_\infty}{U_{\text{sub}}}\right). \quad (5.1)$$

In the third term, $g_{\text{eff}}(U_\infty)$ is the (dimensionless) effective coupling (2.8) of the five-dimensional gauge theory evaluated at the cut-off scale U_∞ . Hence the α' expansion on the string worldsheet produces an expansion in inverse powers of the coupling g_{eff} from the gauge theory perspective, rather than the $1/N_c$ expansion as produced by α' -corrections to the supergravity action — a similar observation was made about the thermal quark diffusion constant in [35]. It is interesting that the energy scale $U_\infty/g_{\text{eff}}(U_\infty)$ appearing in the second subtraction is the supergravity energy scale associated with u_∞ [18]. This is a natural energy scale to appear here since fluctuations on the worldsheet are contributing at this order [26].

In section 4, we explicitly calculated the expectation value of the nonlocal fermion bilinear. This expectation value characterizes the chiral condensate in this holographic model. As this holographic construction does not permit the construction of a local fermion bilinear, this expectation value is the best order parameter to characterize the spontaneous breaking of chiral symmetry. Our explicit calculations yield the result given in (4.21). We note that (2.6) and (4.4) can be used to express the pre-factor of the Polyakov term as $\lambda_5/L \sim g_{\text{eff}}(U_0)$, up to numerical factors, where $g_{\text{eff}}(U_0)$ is the effective coupling evaluated

at the chiral symmetry breaking scale U_0 . The dependence of $\langle \mathcal{O} \rangle$ on L/r_4 illustrated in figure 5 describes the intricate interplay of the supersymmetry breaking (or confinement) and chiral symmetry breaking scales in determining the expectation value. Of course, in the absence of supersymmetry breaking, the result in the extremal background (4.20) is independent of L/r_4 [10]. As figure 5 also illustrates, $\langle \mathcal{O} \rangle$ approaches this supersymmetric result in the limit $L/r_4 \rightarrow 0$.

Studying the theory at finite temperature in this regime, one finds that the chiral symmetry breaking and confinement/deconfinement phase transitions are independent [24, 36]. As described in section 2, the chiral symmetry breaking is realized in the gravitational dual by the D8- $\overline{\text{D8}}$ pair joining together to form a smooth U-shaped embedding. The deconfined phase of the gauge theory is represented by replacing the supergravity background by a D4 black hole [15]. The transition between the low-temperature confining phase and the high-temperature deconfined phase occurs when [15]

$$T_{\text{deconf}} = \frac{1}{2\pi r_4}. \tag{5.2}$$

In the deconfined phase, if u_0 is sufficiently large, the tension of the D8-branes can support the U-shaped embedding against gravitational attraction of the black hole, which has the interpretation that the chiral symmetry remains broken in the deconfined phase [24, 36]. Chiral symmetry is not restored until a temperature given by

$$T_{\chi\text{SB}} = \frac{1}{2\pi r_4} \frac{L_c}{L} \simeq \frac{0.154}{L} \tag{5.3}$$

where $L_c = 0.97r_4$. Above this temperature, the gravitational attraction becomes sufficiently large that the D8- $\overline{\text{D8}}$ pair are pulled into the horizon (and the embedding is trivial, i.e., $x^4 = \text{constant}$), as shown in figure 7. The phase structure of the Sakai-Sugimoto model is summarized in figure 8.

In the high temperature phase, where the D8- $\overline{\text{D8}}$ pair is disconnected, the chiral symmetry is restored and so this should be reflected in the expectation value. In particular, beyond the phase transition of [24, 36], one should have $\langle \mathcal{O} \rangle = 0$. This result comes about because, once chiral symmetry is restored, there is no longer a string worldsheet that could contribute to the expectation value. Specifically, with the trivial embedding in the black hole background there is no string worldsheet with a single asymptotic boundary that connects the D8- $\overline{\text{D8}}$ pair. The simplest consistent worldsheet would extend through the ‘Einstein-Rosen’ throat and out to the boundary of the second asymptotic region in the black hole geometry. Hence this worldsheet would be relevant for a correlator of two operators with the second being in the thermofield double of the original gauge theory [37].

In a similar way, these expectation values are useful for characterizing the different phases in theories with many defects, as discussed, e.g., in [38]. Again, one would find $\langle \mathcal{O} \rangle = 0$ for a Wilson line operator connecting two defects which are not dual to a D8- $\overline{\text{D8}}$ pair which are not joined.

As observed at the end of section 3.1, $\langle \mathcal{O} \rangle \propto N_c$ in accord with the standard large N_c counting. In our calculations, we essentially set the number of flavours to one, however, if

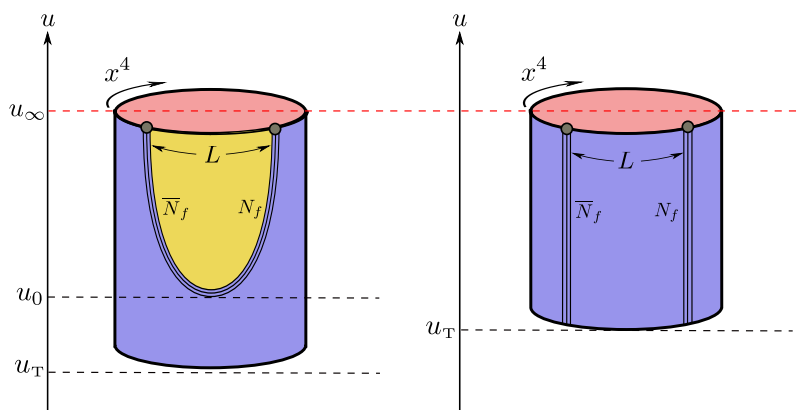


Figure 7: For small L , u_0 is sufficiently large that the D8- $\overline{\text{D8}}$ pair remains outside of the black hole horizon at u_T , as shown on the left. For larger L , the branes fall through the horizon, as shown on the right.

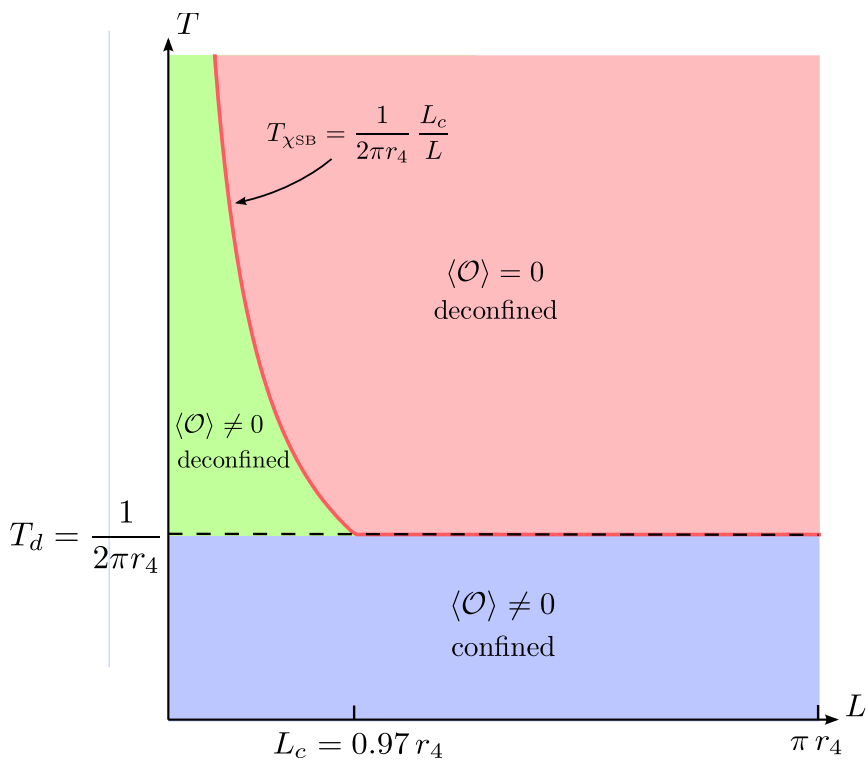


Figure 8: The phase diagram for the Sakai-Sugimoto model. The transition from confinement to deconfinement occurs at $T_d = (2\pi r_4)^{-1}$. Chiral symmetry is also restored at this temperature if $L > 0.97 r_4$. The transitions are independent for $L < 0.97 r_4$, in which case chiral symmetry is restored above $T_{\chi\text{SB}} = .154/L$.

$N_f > 1$ one might anticipate the expectation values would also be proportional to N_f , again reflecting the number of degrees of freedom involved in such a bilinear — e.g., , see [34, 39].

However, in the case where $N_f > 1$, we are implicitly considering the expectation value $\langle \mathcal{O}^{IJ} \rangle$ of an operator with flavour indices I and J for the two fermions. For the smooth embeddings, we would have $\langle \mathcal{O}^{IJ} \rangle \propto \delta^{IJ}$ because consistency requires that the worldsheet start and end on the same brane throughout the embedding. A priori, there is no connection between the ψ_L^I on one defect and the ψ_R^J on the other. So our operator reveals this connection as established by the chiral symmetry breaking. Tracing over the flavour indices would correspond to implicitly summing over the different worldsheets and would produce the factor of N_f mentioned above.

An alternate approach to understanding chiral symmetry breaking in the Sakai-Sugimoto model was considered in [6–9]. There the key element is the open string tachyon that develops between the D8- $\overline{\text{D8}}$ pair when the (proper) distance separating them is small. Chiral symmetry breaking is realized as the condensation of the tachyon, which leads to brane-anti-brane annihilation deep in the IR region, producing the smooth embedding in which the D8- $\overline{\text{D8}}$ pair join. The quark mass and the chiral condensate would be related to the asymptotically growing and decaying modes of the tachyon field. This description and the approach examined in the present paper both consider the physics of open strings stretched between the D8- and $\overline{\text{D8}}$ -branes, so it seems that they must be related. Conceptually, one can think of the tachyon analysis as the second-quantized description of the relevant open string physics while the worldsheet procedure [10–12] considered above is the first-quantized description of essentially the same physics. Of course, it would be interesting to make this connection more precise. This naturally calls for a proper quantization of (open) strings in the supergravity background of the D4-brane throat. A more accessible route may be to examine the D8-brane embeddings for a nonvanishing quark mass, following the suggestion of [10] to include the Polyakov action for the instantonic worldsheet as part of the action for the D8- $\overline{\text{D8}}$ pair. One could then consider the dependence of $\langle \mathcal{O} \rangle$ on m_q and compare with the results given in [7–9].

Acknowledgments

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A. Fluctuation determinant

In this appendix we will evaluate the contribution from fluctuations for rectangular worldsheet discussed in section 3.1. Since α' is a loop-counting parameter on the string worldsheet, the one-loop fluctuation determinant will contribute at the same order as the Fradkin-Tseytlin term (3.7), considered in the main text. The primary result here is to

show that the fluctuation determinant contributes no additional UV divergences to the Wilson line calculations, at this order. Since the UV behaviour is universal, this result applies for all of the Wilson line calculations considered in the present paper. A similar analysis of worldsheet fluctuations for a standard closed Wilson line in the D4-brane background has been performed in [40]. To begin the calculation, we must expand the worldsheet action to quadratic order about a classical solution, X^i . Working with the Green-Schwarz formalism,⁵ this yields

$$I = I_B + I_F + I_{\text{ghosts}} , \tag{A.1}$$

where

$$I_B = \frac{1}{4\pi} \int d^2\sigma \sqrt{g} \left(G_{ij} D_\alpha \xi^i D_\beta \xi^j g^{\alpha\beta} + R_{ik_1k_2j} \xi^{k_1} \xi^{k_2} \partial_\alpha X^i \partial_\beta X^j g^{\alpha\beta} + R^{(2)} \left[\ell_s D_i \Phi \xi^i + \frac{1}{2} \ell_s^2 D_i D_j \Phi \xi^i \xi^j \right] \right) , \tag{A.2}$$

$$I_F = \frac{i}{4\pi} \int d^2\sigma \bar{\Theta} [(\sqrt{-h} h^{\alpha\beta} + \epsilon^{\alpha\beta} \Gamma^{11}) \Gamma_\alpha \mathcal{D}_\beta] \Theta , \tag{A.3}$$

and

$$I_{\text{ghosts}} = \frac{1}{2} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \left(g^{\gamma\delta} \nabla_\gamma \epsilon_\alpha \nabla_\delta \epsilon_\beta - \frac{1}{2} R^{(2)} \epsilon_\alpha \epsilon_\beta \right) . \tag{A.4}$$

In the above, i labels a spacetime index which we will split in what follows as a labeling a S^4 direction and μ labeling the remaining transverse directions. The worldsheet directions will be labeled by lower Greek indices. The other quantities are specified by [40]

$$D_\alpha \xi^j = \partial_\alpha X^i (\partial_i \xi^j + \Gamma_{il}^j \xi^l) , \tag{A.5}$$

$$\mathcal{D}_i = \partial_i + \frac{1}{4} \omega_{iab} \Gamma^{ab} - \frac{1}{8 \cdot 4!} e^\phi F_{abcd} \Gamma^{abcd} \Gamma_i \equiv \partial_i + M_i , \tag{A.6}$$

where a, b, c, d are tangent space indices. The κ -symmetry transformation for the GS fermions is given by

$$\delta_\kappa \Theta = \left(1 - \frac{\epsilon^{\alpha\beta}}{2\sqrt{-g}} \Gamma_{\alpha\beta} \Gamma^{11} \right) \kappa . \tag{A.7}$$

Note that for simplicity in the following calculations, we will set $R = 1$ in the supergravity background. We will also drop the dilaton terms in what follows since they are subleading.

Fermion contributions. We will use the zehnbeins

$$e^\mu = u^{3/4} dx^\mu , \quad \mu = 1 \dots 4 \tag{A.8}$$

$$e^0 = u^{3/4} dt , \quad e^5 = u^{-3/4} du , \tag{A.9}$$

$$e^6 = u^{1/4} d\psi , \quad e^7 = u^{1/4} \cos \psi d\chi , \tag{A.10}$$

$$e^8 = u^{1/4} \cos \psi \sin \chi d\phi_1 , \quad e^9 = u^{1/4} \cos \psi \cos \chi d\phi_2 . \tag{A.11}$$

⁵One might question whether or not the Fradkin-Tseytlin term (3.7) is to be added in the worldsheet action of the Green-Schwarz string. While the classical action does not couple to the dilaton, this interaction is still necessary at the quantum level to preserve the conformal and κ symmetry of Green-Schwarz string, just as in the bosonic case (and also to have proper effective string coupling dependence of string loops).

In the following we will sometimes use the notation $\mu_6 = 1, \mu_7 = \cos\psi, \mu_8 = \cos\psi \sin\chi, \mu^9 = \cos\psi \cos\chi$. The RR field strength in tangent space is given by

$$F_{6789} = \frac{3}{u}. \quad (\text{A.12})$$

In the case at hand, $x^4 = \sigma_0$ and $u = \sigma_1$ where σ_0, σ_1 are worldsheet variables. Using the pullback metric for the Euclidean worldsheet and noting that $\sqrt{-g} = i\sqrt{|g|}$ we get

$$I_F = \frac{i}{4\pi} \int d^2\sigma \left(\bar{\Theta} \Gamma^{11} (\Gamma_u \mathcal{D}_0 - \Gamma_{x^4} \mathcal{D}_1) \Theta - i \bar{\Theta} (\sigma_1^{3/2} \Gamma_u \mathcal{D}_1 - \sigma^{-3/2} \Gamma_{x^4} \mathcal{D}_0) \Theta \right). \quad (\text{A.13})$$

Here we have absorbed α' into the fluctuations. Using

$$\Gamma_u M_0 - \Gamma_\tau M_1 = -\frac{3}{8} u^{-1/4} \Gamma_4 + \frac{3}{4} u^{-1/4} \tilde{\Gamma} \Gamma_{45}, \quad (\text{A.14})$$

$$\Gamma_u M_1 = -\frac{3}{8} u^{1/4} \tilde{\Gamma}, \quad (\text{A.15})$$

$$\Gamma_\tau M_0 = \frac{3}{8} u^{5/4} - \frac{3}{8} u^{5/4} \tilde{\Gamma}, \quad (\text{A.16})$$

with $\tilde{\Gamma} = \Gamma^{6789}$ in tangent space we get

$$I_F = \frac{i}{4\pi} \int d^2\sigma \left(u^{-3/4} \bar{\Theta} (\Gamma^{11} \Gamma_5 - i \Gamma_4) \partial_0 \Theta + u^{3/4} \bar{\Theta} (-i \Gamma_5 - \Gamma^{11} \Gamma_4) \partial_1 \Theta - \frac{3u^{-1/4}}{8} \bar{\Theta} (\Gamma^{11} \Gamma_4 - i \Gamma_5) \Theta + \frac{3u^{-1/4}}{4} \bar{\Theta} (i \tilde{\Gamma} + \Gamma^{11} \tilde{\Gamma} \Gamma_{45}) \Theta \right). \quad (\text{A.17})$$

We will fix κ -symmetry with the following: First split $\Theta = \theta_1 + \theta_2$. Then choose

$$\Gamma_5 \theta^1 = i \Gamma_4 \theta^1, \quad \Gamma_5 \theta^2 = -i \Gamma_4 \theta^2, \quad (\text{A.18})$$

which leads to after redefining $\hat{\theta}^{1,2} = u^{3/8} \theta^{1,2}$ and defining $\partial_\pm = \mp i \sigma^{-3/2} \partial_0 + \partial_1$

$$I_F = \frac{i}{2\pi} \int d^2\sigma \left(\bar{\hat{\theta}}^1 \Gamma_4 \partial_+ \hat{\theta}^1 - \bar{\hat{\theta}}^2 \Gamma_4 \partial_- \hat{\theta}^2 + \frac{3i}{4\sigma_1} (\bar{\hat{\theta}}^1 \tilde{\Gamma} \hat{\theta}^2 + \bar{\hat{\theta}}^2 \tilde{\Gamma} \hat{\theta}^1) \right). \quad (\text{A.19})$$

Now choosing the gamma matrices Γ_a such that

$$\Gamma_0 = \tau_2 \otimes 1, \Gamma_1 = \tau_1 \otimes 1, \Gamma_A = \tau_3 \otimes \gamma_A \quad (\text{A.20})$$

with γ_A being Euclidean Dirac matrices in 8 dimensions and splitting the θ^I 's into two Euclidean Majorana-Weyl fermions of opposite chiralities S, \tilde{S} we get

$$I_F = \frac{i}{2\pi} \int d^2\sigma \left(S \partial_+ S - \tilde{S} \partial_- \tilde{S} + \frac{3}{4\sigma_1} (S \tilde{\gamma} \tilde{S} - \tilde{S} \tilde{\gamma} S) \right). \quad (\text{A.21})$$

The squared equations of motion following from the above are:

$$\left(\partial_0^2 + \sigma_1^{3/2} \partial_1 \sigma_1^{3/2} \partial_1 - \frac{\sqrt{\sigma_1}}{2} (i \partial_0 + \sigma^{3/2} \partial_1) - \frac{9}{16} \sigma_1 \right) S = 0, \quad (\text{A.22})$$

$$\left(\partial_0^2 + \sigma_1^{3/2} \partial_1 \sigma_1^{3/2} \partial_1 - \frac{\sqrt{\sigma_1}}{2} (-i \partial_0 + \sigma^{3/2} \partial_1) - \frac{9}{16} \sigma_1 \right) \tilde{S} = 0. \quad (\text{A.23})$$

Combining S, \tilde{S} into a worldsheet spinor $\Psi = (S, \tilde{S})$, the equations of motion for the fermions then takes on the form

$$\left(\partial_0^2 + \sigma_1^{3/2} \partial_1 \sigma_1^{3/2} \partial_1 - \frac{9\sigma_1}{16} + \frac{3\sigma_1}{8} \tau^1 \tilde{\gamma} \right) \Psi = 0. \quad (\text{A.24})$$

Here τ^1 represents the Pauli matrix $((0, 1), (1, 0))$. In first order perturbation theory, the term proportional to τ_1 will not contribute. Since S and \tilde{S} combine to form a worldsheet spinor, we have 8 massive fermions satisfying the above equation.⁶

Boson contributions. In the second line of (A.2), we have included the contributions of the Fradkin-Tseytlin term (3.7). However, these two terms come with explicit factors of the string length ℓ_s which reflects the fact that they would only contribute in a two-loop calculation of the fluctuation determinant. Therefore we ignore these last two contributions in the following calculation. The quadratic order action for the bosons is given by

$$I_B = \frac{1}{4\pi} \int d^2\sigma \frac{1}{\sigma_1^{3/2}} \left(\dot{\xi}_u^2 + \sigma_1^3 \dot{\xi}_u'^2 + \frac{15\sigma_1}{16} \hat{\xi}_u^2 + \dot{\xi}_{x^4}^2 + \sigma_1^3 \dot{\xi}_{x^4}'^2 + \frac{15\sigma_1}{16} \hat{\xi}_{x^4}^2 - 24\sqrt{\sigma_1} (\hat{\xi}_{x^4} \dot{\xi}_u - \dot{\xi}_u \hat{\xi}_{x^4}) \right. \\ \left. + \dot{\xi}_\mu^2 + \sigma_1^3 \dot{\xi}_\mu'^2 + \frac{15\sigma_1}{16} \hat{\xi}_\mu^2 + \mu_a^2 \left(\dot{\xi}_a^2 + \sigma_1^3 \dot{\xi}_a'^2 + \frac{3\sigma_1}{16} \hat{\xi}_a^2 \right) \right). \quad (\text{A.25})$$

Here we have defined $\hat{\xi}_{x^4} = u^{3/4} \xi_{x^4}$, $\hat{\xi}_u = u^{-3/4} \xi_u$, $\hat{\xi}_\mu = u^{3/4} \xi_\mu$, $\hat{\xi}_a = u^{1/4} \xi_a$. Thus the mass terms are at $O(1/\sqrt{\sigma_1})$. Hence we now have $\hat{\xi}_u, \hat{\xi}_{x^4}, \hat{\xi}_\mu$ satisfying

$$\left(\partial_0^2 + \sigma_1^{3/2} \partial_1 \sigma_1^{3/2} \partial_1 - \frac{15\sigma}{16} \right) \hat{\xi} = 0, \quad (\text{A.26})$$

while $\hat{\xi}_a$ satisfy

$$\left(\partial_0^2 + \sigma_1^{3/2} \partial_1 \sigma_1^{3/2} - \frac{3\sigma}{16} \right) \hat{\xi} = 0, \quad (\text{A.27})$$

Ghost contributions. The ghost action works out to be

$$I_{\text{ghosts}} = \frac{1}{2} \int d^2\sigma \frac{1}{\sigma_1^{9/2}} \left(\dot{\hat{\epsilon}}_1^2 + \sigma_1^3 \dot{\hat{\epsilon}}_1'^2 + \frac{15\sigma}{16} \hat{\epsilon}_1^2 + \sigma_1^6 \left(\dot{\hat{\epsilon}}_2^2 + \sigma_1^3 \dot{\hat{\epsilon}}_2'^2 + \frac{15\sigma}{16} \hat{\epsilon}_2^2 \right) - 24\sigma^{7/2} (\hat{\epsilon}_2 \dot{\hat{\epsilon}}_2 - \dot{\hat{\epsilon}}_2 \hat{\epsilon}_2) \right), \quad (\text{A.28})$$

with $\hat{\epsilon}_1 = \sigma_1^{3/4} \epsilon_1$ and $\hat{\epsilon}_2 = \sigma_1^{-3/4} \epsilon_2$. The ghosts satisfy

$$\left(\partial_0^2 + \sigma_1^{9/2} \partial_1 \sigma_1^{-3/2} \partial_1 - \frac{15\sigma_1}{16} \right) \hat{\epsilon}_1 = 0, \quad (\text{A.29})$$

$$\left(\partial_0^2 + \sigma_1^{-3/2} \partial_1 \sigma_1^{9/2} \partial_1 - \frac{15\sigma_1}{16} \right) \hat{\epsilon}_2 = 0. \quad (\text{A.30})$$

⁶Otherwise naively it would appear that there are 8 S 's and 8 \tilde{S} 's giving 16 fermions which would lead to the wrong counting.

Final result. The result for the partition function after the above laborious calculation is

$$\frac{\det^{8/2}(\partial_t^2 + \sigma^{3/2} \partial_\sigma \sigma^{3/2} \partial_\sigma - \frac{9}{16} \sigma) \det^{1/2}(\partial_t^2 + \sigma^{9/2} \partial_\sigma \sigma^{-3/2} \partial_\sigma - \frac{15}{16} \sigma) \det^{1/2}(\partial_t^2 + \sigma^{-3/2} \partial_\sigma \sigma^{9/2} \partial_\sigma - \frac{15}{16} \sigma)}{\det^{6/2}(\partial_t^2 + \sigma^{3/2} \partial_\sigma \sigma^{3/2} \partial_\sigma - \frac{15}{16} \sigma) \det^{4/2}(\partial_t^2 + \sigma^{3/2} \partial_\sigma \sigma^{3/2} \partial_\sigma - \frac{3}{16} \sigma)} \quad (\text{A.31})$$

The evaluation of the determinant exactly is in general a very hard problem [41]. In this case we note that (A.31) can be rewritten as

$$\frac{\det^{8/2}(\partial_t^2 + \partial_x^2 - \frac{f}{x^2}) \det^{1/2}(\partial_t^2 + \partial_x^2 - \frac{g_1}{x^2}) \det^{1/2}(\partial_t^2 + \partial_x^2 - \frac{g_2}{x^2})}{\det^{6/2}(\partial_t^2 + \partial_x^2 - \frac{b_1}{x^2}) \det^{4/2}(\partial_t^2 + \partial_x^2 - \frac{b_2}{x^2})}, \quad (\text{A.32})$$

where $x \equiv -\frac{2}{\sqrt{\sigma}}$ and $f = 9/4, g_1 = 39/4, g_2 = 63/4, b_1 = 15/4, b_2 = 3/4$. Then each of the operators featuring in the determinant can be written as [41]

$$D = \partial_x^2 - \frac{(l-1/2)(l+1/2)}{x^2}, \quad (\text{A.33})$$

so that we can write the determinant as $\prod \lambda_n$ where λ_n is given by

$$\left(-D + \frac{4m^2\pi^2}{L^2} \right) \phi_n = -\lambda_n \phi_n. \quad (\text{A.34})$$

The function ϕ_n satisfies Dirichlet boundary conditions, namely $\phi_n(x=0) = 0, \phi_n(x=x_0) = 0$. Here x_0 is related to u_0 through $x_0 = -2/\sqrt{u_0}$. The solution to (A.34) with the Dirichlet boundary conditions are known to be Bessel functions $\sqrt{x}J_l(x), \sqrt{x}Y_l(x)$. Since both $\pm l$ are allowed, we will choose $\sqrt{x}J_l(x)$ and $\sqrt{x}Y_{-l}(x)$ to be the independent solutions. Then imposing the boundary condition at $x = x_0$ we have

$$J_l(\omega_n x_0) = 0, \quad \omega_n^2 = \left(\frac{j_{l,n}}{x_0} \right)^2 = \lambda_n - \frac{4m^2\pi^2}{L^2}, \quad (\text{A.35})$$

$$Y_{-l}(\omega_n x_0) = 0, \quad \omega_n^2 = \left(\frac{y_{-l,n}}{L} \right)^2 = \lambda_n - \frac{4m^2\pi^2}{L^2}, \quad (\text{A.36})$$

so that ω_n 's are related to the zeros $j_{l,n}, y_{-l,n}$ of the Bessel functions. Then the determinant can be written using the formula

$$\sinh x = x \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{k^2\pi^2} \right), \quad (\text{A.37})$$

as

$$\mathcal{D} = \mathcal{D}_j \mathcal{D}_y, \quad (\text{A.38})$$

where

$$\mathcal{D}_j = \prod \frac{\sinh^4 \frac{j_{l_f, n_f} L}{2|x_0|} \sinh^{1/2} \frac{j_{l_{g_1}, n_{g_2}} L}{2|x_0|} \sinh^{1/2} \frac{j_{l_{g_2}, n_{g_2}} L}{2|x_0|}}{\sinh^3 \frac{j_{l_{b_1}, n_{b_1}} L}{2|x_0|} \sinh^2 \frac{j_{l_{b_2}, n_{b_2}} L}{2|x_0|}}. \quad (\text{A.39})$$

$$\mathcal{D}_y = \prod \frac{\sinh^4 \frac{y_{-l_f, n_f} L}{2|x_0|} \sinh^{1/2} \frac{y_{-l_{g_1}, n_{g_2}} L}{2|x_0|} \sinh^{1/2} \frac{y_{-l_{g_2}, n_{g_2}} L}{2|x_0|}}{\sinh^3 \frac{y_{-l_{b_1}, n_{b_1}} L}{2|x_0|} \sinh^2 \frac{y_{-l_{b_2}, n_{b_2}} L}{2|x_0|}}. \quad (\text{A.40})$$

Now we want to get the large n asymptotics of this function. We use the useful identity [42] that the large zeros of the Bessel function behave as

$$j_{\nu,n} \cos \alpha - y_{\nu,n} \sin \alpha \approx \left(n + \frac{\nu}{2} - \frac{1}{4}\right)\pi - \alpha - \frac{4\nu^2 - 1}{8\left[\left(n + \frac{\nu}{2} - \frac{1}{4}\right)\pi - \alpha\right]} + \dots, \quad (\text{A.41})$$

to get

$$\log \mathcal{D} \approx 0.09\pi \sum_n \left(8\left(n \pm \frac{1}{2}\sqrt{\frac{5}{2}} \mp \frac{1}{4}\right) + \left(n \pm \frac{1}{2}\sqrt{10} \mp \frac{1}{4}\right) + \left(n \pm 2 \mp \frac{1}{4}\right) - 6\left(n \pm 1 \mp \frac{1}{4}\right) - 4\left(n \pm \frac{1}{2} \mp \frac{1}{4}\right) \right), \quad (\text{A.42})$$

using which the leading divergence cancels. The subleading terms arise from $O(1/n^2)$ terms which lead to a finite result at $O(1/u_\infty^{9/2})$. The exact formula (A.38) allows us in principle to extract this finite number although we will not attempt it here, as this contribution would vanish in the relevant limit $u_\infty \rightarrow \infty$. Hence our key result is that in the fluctuation determinant (A.31) is in fact precisely 1 in this limit.

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